Kaprekar’s transformations.
Part II – numerical results and intriguing corollaries

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Abstract—This paper is a continuation of our previous paper [Part I, ibidem]. In this study we present many new results in the subject of minimal cycles (including the fixed points) of the so-called Kaprekar’s transformations. We formulate also some conjectures. Moreover, we discuss here all minimal cycles of the subject of minimal cycles (including the fixed points) of the so-called Kaprekar’s transformations. We formulate also some conjectures concerning each of the investigated cycles. The other cases for \( n = 16 \) – 18, because of the permissible length of the paper, are omitted here.

I. INTRODUCTION

In Part I of this elaboration (see [1]) we have introduced the definitions of the so-called Kaprekar’s transformations \( T_n \):

\[
T_n : \{0\} \cup \{ \alpha : 10^{n-1} \leq \alpha < 10^n \} \rightarrow \{0\} \cup \{ \alpha : 10^{n-1} \leq \alpha < 10^n \}
\]

\[
T_n(\alpha) = \sum_{k=1}^{n}(a_k - a_{n-k+1})10^{k-1}
\]

for every \( \alpha, n \in \mathbb{N} \), \( 10^{n-1} \leq \alpha < 10^n \), where

\[
0 \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq 9,
\]

denote all digits of decimal expansion of number \( \alpha \) ordered in nondecreasing sequence and \( T_n(0) = 0 \). We have also described the orbits of maps \( T_n \) for \( n = 3, 4, \ldots, 7 \). Furthermore, in Part I many new concepts and characteristics of the minimal cycles of general transformations \( F : X \rightarrow X \), where \( X \) is a finite set, have been proposed. All of them will be used in this part of our paper and applied for the Kaprekar’s transformations \( T_n, n \in \mathbb{N} \).

Moreover, in this part of our paper we intend to present firstly the collection of absolutely new facts discovered by observing the, numerically obtained, orbits of operators \( T_n \) for \( n \leq 18 \). Next we will compile in tables the detailed descriptions of the minimal cycles of operators \( T_n \) for \( n \leq 15 \) (that is, we will give many individual pieces of information concerning each of the investigated cycles). The other cases for \( n = 16 \) – 18, because of the permissible length of the paper, are omitted here.

II. FACTS BASED ON THE NUMERICAL RESULTS

Let us present now several essential facts in the subject of Kaprekar’s transformations which we have deduced by analyzing the numerically obtained minimal cycles of operators \( T_n \) for \( n \leq 18 \). We will also formulate some conjectures concerning the cycles of Kaprekar’s transformations.

Fact 1. Numbers appearing in the orbits of transformations \( T_n \) correspond with the partitions of number \( \frac{9}{2} \times n \) into \( n \) digits, except the following \( n = 3k \)-digit numbers being the Kaprekar’s constants of order \( 3k \) with the sum of digits equal to \( 18k \):

\[
495, 549945, 554999445, \ldots, \, 5, \ldots, 5 \quad 49 \ldots 9 \quad 4 \ldots 4 \quad 5.
\]

(k–1) digits \( k \) digits \( k \) digits \( (k–1) \) digits

The following theorem and the respective conclusions constitute the theoretical grounds of the described above properties of the orbits of transformations \( T_n \).

Theorem 1.

a) Let \( a \in \mathbb{N} \) be a \( 2n \)-digit number composed of digits

\[
0 \leq a_1 \leq a_2 \leq \ldots \leq a_{2n} \leq 9
\]

and suppose that

\[
a_{n-k-1} < a_{n-k} = a_{n-k+1} = \ldots = a_{n+l} < a_{n+l+1}
\]

for some \( k, l \in \mathbb{N}_0 \).

If \( k \geq l \), then the sum of digits of number \( T_{2n}(a) \) is equal to \( 9 \times (n+l) \). Otherwise, this sum is equal to \( 9 \times (n+k) \).

b) Let \( a \in \mathbb{N} \) be a \( (2n+1) \)-digit number composed of digits

\[
0 \leq a_1 \leq a_2 \leq \ldots \leq a_{2n+1} \leq 9
\]

and suppose that

\[
a_{n-k} < a_{n-k+1} = a_{n-k+2} = \ldots = a_{n+l+1} < a_{n+l+2}
\]

for some \( k, l \in \mathbb{N}_0 \).

If \( k \geq l \), then the sum of digits of number \( T_{2n+1}(a) \) is equal to \( 9 \times (n+l+1) \), whereas if \( k < l \), then the sum of digits of number \( T_{2n+1}(a) \) is equal to \( 9 \times (n+k+1) \).

Proof:

ad a) Let us notice that the following decimal expansions of \( T_{2n}(a) \) can be obtained

\[
T_{2n}(a) = \begin{cases} (a_{2n} - a_1)(a_{2n-1} - a_2) \ldots (a_{n+k+1} - a_{n-k} - 1) \\ \times (9 + a_{n+k} - a_{n-k}) \ldots (9 + a_2 - a_{2n-1}) \\ \times (10 + a_1 - a_{2n}) \end{cases}, \quad \text{if } l > k,
\]

\[
T_{2n}(a) = \begin{cases} (a_{2n} - a_1)(a_{2n-1} - a_2) \ldots (a_{n+k+1} - a_{n-k} - 1) \\ \times (9 + a_{n+l} - a_{n-l}) \ldots (9 + a_2 - a_{2n-1}) \\ \times (10 + a_1 - a_{2n}) \end{cases}, \quad \text{if } k \geq l,
\]
which implies the assertion.

ad b) The proof runs in similar way as in case of item a). ■

Corollary 1. If \( a \in \mathbb{N} \) is a \( n \)-digit number then the sum of digits of number \( T_n(a) \) is not lower than the number \( 9 \times \left\lfloor \frac{n}{2} \right\rfloor \).

Corollary 2. If \( a \in \mathbb{N} \) is a number possessing different digits in the decimal expansion then the sum of digits of number \( T_n(a) \) is equal to \( 9 \times \left\lfloor \frac{n}{2} \right\rfloor \).

Conjecture 1. Sum of digits of the numbers belonging to the given orbit of operator \( T_n \), where \( n \in \mathbb{N} \), except the two-element orbit of operator \( T_n \), is the same.

Remark 1. The lowest number \( n \), for which there exist two different orbits (two different orbits possessing at least two elements) of operator \( T_n \) composed of the numbers with different sums of digits, is equal to 6 (is equal to \( n = 16 \), respectively).

Remark 2. Numbers belonging to the orbits of operator \( T_{2n+1} \) possess in their decimal expansion the middle digit equal to 9.

Fact 2. Let \( a_1a_2\ldots a_n \) be an \( n \)-digit number belonging to some orbit of transformation \( T_n \), \( n \in \mathbb{N} \). Then the sequence, henceforward called as the digit type of element \( a_1a_2\ldots a_n \) of the given cycle, defined in the following way

\[
a_1+a_{n+1}, a_2+a_{n+2}, a_3+a_{n+3}, \ldots, a_{n}+a_1,
\]

is equal to

\[
\begin{align*}
10, 9, \ldots, 9, 8, 9, \\
&\text{ (}k-1\text{-times)}
\end{align*}
\]

if \( n = 2k+1 \), \( k = 1, 2, \ldots \), and

\[
\begin{align*}
10, 9, \ldots, 9, 8, \\
&\text{ (}k-2\text{-times)}
\end{align*}
\]

if \( n = 2k \), \( k = 2, 3, \ldots \). In both cases the equality holds independently on number \( a_1a_2\ldots a_n \), except the following numbers:

(i) the Kaprekar’s constants of order \( n = 3k \):

\[
\begin{array}{c}
5 \ldots 5 \\
\text{(}k\text{-times)}
\end{array}
\]

for which the respective sequence of sums has the form

\[
\begin{array}{c}
10, 9, \ldots, 9, 8, 18, \ldots, 18, \\
\text{(}k\text{-times)}
\end{array}
\]

Let us notice that if we correct the above sequence in the following way (we shift the units similarly as in the addition operation):

\[
\begin{array}{c}
10, 9, \ldots, 9, 8, 18, \ldots, 18, 9, \\
\text{(}k-1\text{-times)}
\end{array}
\]

then we obtain the sequence

\[
\begin{array}{c}
10, 9, \ldots, 9, 8, 9, \\
\text{(}k\text{-times)}
\end{array}
\]

which is “compatible” either with (1), if \( k \) is odd, or with (2), if \( k \) is even.

(ii) the numbers belonging to the single 2-element orbit \( \{59355, 59994\} \) of operator \( T_5 \), where the respective sequences of sums are of the forms 10, 8, 9 and 9, 18, 9, but we get

\[
\begin{array}{c}
9, 1, 8, 9, 9, 10, 9, 8, 9, \\
\text{(2-}\text{times)}
\end{array}
\]

(iii) the numbers belonging to the single 2-element orbit \( \{876421997755322, 8765431997654322\} \) of operator \( T_{18} \), where both sequences of sums are of the form 10, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 8, 18, but we obtain

\[
\begin{array}{c}
10, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 8, \\
\text{(2-}\text{times)}
\end{array}
\]

which is compatible with (2).

Fact 3. We have noticed that for every \( n = 10, 12, \ldots, 18 \) the operator \( T_n \) possesses the even number of 3-element cycles and, moreover, the difference between the numbers of 3-element cycles of \( T_n \) possessing the orbit types \( (1, 3, 2) \) and \( (1, 2, 3) \), respectively, is equal to 0 for \( n = 10, 12 \) and \( \frac{n-12}{2} \) for \( n = 14, 16, 18 \). The orbit type of all 7-element cycles of \( T_n \), \( n \leq 18 \), is the same and is equal to \( (1, 5, 3, 4, 6, 7, 2) \).

Fact 4 (Kaprekar’s constants). We have observed that each Kaprekar’s constant of order \( n \leq 18 \) generates the sequence of extensions of decimal expansions remaining the Kaprekar’s constants (of the respectively higher order). For example, we have

- \( 63 \ldots 3176 \ldots 64 \) are the Kaprekar’s constants of order \( k \)-times \( (k+4) \)-times for every \( k = 0, 1, 2, \ldots \).

Sketch of the proof: We have

\[
\begin{array}{c}
76 \ldots 643 \ldots 31 \ldots 34 \ldots 67 = 63 \ldots 3176 \ldots 64 \\
\text{ (}k\text{-times)}
\end{array}
\]

\[
\begin{array}{c}
9 \ldots 98750842 \ldots 0 \ldots 01 \ldots \ldots \ldots \ldots \ldots \ldots \\
\text{ (}k\text{-times)}
\end{array}
\]

\[
\begin{array}{c}
9753 \ldots 3086 \ldots 421 \ldots \ldots \ldots \ldots \ldots \ldots \\
\text{ (}k\text{-times)}
\end{array}
\]

\[
\begin{array}{c}
9 \ldots 975308642 \ldots 0 \ldots 01 \ldots \ldots \ldots \ldots \ldots \ldots \\
\text{ (}k\text{-times)}
\end{array}
\]

\[
\begin{array}{c}
8643 \ldots 31976 \ldots 6532 \ldots \ldots \ldots \ldots \ldots \ldots \\
\text{ (}k\text{-times)}
\end{array}
\]

\[
\begin{array}{c}
10, 9 \ldots 9, 8, 9, \\
\text{(}k\text{-times)}
\end{array}
\]

which is compatible with (2).
Remark 3. The Q-Kaprekar's transformations $Q_n$, defined in the last section of Part I, possess the same property as above for their fixed points. For example, the number

$$5 \cdots 5 4 \cdots 9 \cdots 9 \cdots 4 \cdots 15 \quad (k\text{-times})$$

is the fixed point of transformation $Q_{3k+3}$ for every $k = 1, 2, \ldots$. the number

$$663 \cdots 3 086 \cdots 6 572 \quad (k\text{-times})$$

is the fixed point of $Q_{2k+6}$ for every $k = 0, 1, 2, \ldots$ and, at last, the number

$$9 \cdots 9 7508420 \cdots 0 1 \quad (k+1\text{-times})$$

is the fixed point of $Q_{2k+8}$ for every $k = 1, 2, \ldots$.

Fact 5. We suppose that, similarly like in case of the Kaprekar's constants, all orbits of operators $T_n$ with the odd number of elements possess their "extensions", that is they generate the infinite sequences of orbits of the Kaprekar's operators preserving the number of elements of the initial orbit. Whereas, despite of the insistent efforts we did not manage to get such extension (in the similar style as in case of the orbits presented below) for any orbit having the even number of elements.

The Kaprekar's transformation $T_{2(k+4)}$, for $k = 0, 1, \ldots, 5$, possesses $A140226(k)$ (equal to $4k(11 + k^2)$ for $k \geq 1$) of 3-element minimal cycles ($A140226$ in notation of the Sloane's OEIS).

Furthermore, transformation $T_{2(k+4)}$, for each $k = 0, 1, \ldots$ possesses the following 3-element minimal cycle

$$\begin{align*}
643 \cdots 3 086 & \cdots 6 654, \\
383 & \cdots 3 2087 \cdots 6 662, \\
8653 & \cdots 3 266 \cdots 6 432.
\end{align*}$$

For $k = 0$ it is the single 3-element minimal cycle of the respective Kaprekar's transformation.

The other examples of 3-element minimal cycles of maps $T_{2k+6}$, $T_{2k+10}$, $T_{2k+13}$, are the following:

$$\begin{align*}
(87 \cdots 73 & \cdots 3 32087 \cdots 6 622 \cdots 2, \\
6435 & \cdots 5 3 \cdots 3 266 \cdots 6 4 \cdots 4 432, \\
9753 \cdots 3 10886 & \cdots 6 421, 97575 \cdots 3 086 \cdots 6 4221, \\
97553 & \cdots 3 086 \cdots 6 4421).
\end{align*}$$

respectively, for every $k = 0, 1, 2, \ldots$ Every Kaprekar's transformation $T_{2k+13}$, for $k = 0, 1, 2, \ldots$ possesses the following 5-element minimal cycle

$$(8643 \cdots 3 20987 \cdots 6 \cdots 6 532, 96643 \cdots 3 1976 \cdots 6 \cdots 6 5331, 88433 \cdots 3 1976 \cdots 6 \cdots 6 5612, 87643 \cdots 3 1976 \cdots 6 \cdots 6 5322, 86543 \cdots 3 1976 \cdots 6 \cdots 6 5432).$$

For $k = 0$ it is the single 5-element minimal cycle of the respective Kaprekar's transformation.

Next, the transformation $T_{2(k+13)}$, for every $k = 0, 1, 2, \ldots$, has also two following 5-element minimal cycles (all these cycles possess the same orbit type equal to $(1, 4, 5, 3, 2)$ and $(1, 4, 2, 5, 3)$, respectively):

$$\begin{align*}
(86543 & \cdots 3 20987 \cdots 6 \cdots 6 5432, 96643 \cdots 3 20987 \cdots 6 \cdots 6 5331, 98643 \cdots 3 1976 \cdots 6 \cdots 6 6511, 88743 \cdots 3 1976 \cdots 6 \cdots 6 5212, 876543 \cdots 3 1976 \cdots 6 \cdots 6 54322, \\
8(8764 \cdots 3 20987 & \cdots 6 \cdots 6 54331, 966543 \cdots 3 1976 \cdots 6 \cdots 6 54331, 88433 \cdots 3 20987 \cdots 6 \cdots 6 5612, 976643 \cdots 3 1976 \cdots 6 \cdots 6 5321, 88543 \cdots 3 1976 \cdots 6 \cdots 6 5412).\end{align*}$$

For $k = 0$ three above 5-element minimal cycles are the only 5-element minimal cycles of $T_{2k+13}$.

The example of 7-element cycle of map $T_{2k+6}$, for every $k = 0, 1, 2, \ldots$, is the following (which possesses the orbit type
equal to \((1, 5, 3, 4, 6, 7, 2)\): 
\[
\begin{align*}
(43\ldots 320876\ldots 66, & \quad 853\ldots 3176\ldots 642, \\
753\ldots 3086\ldots 643, & \quad 843\ldots 3086\ldots 652, \\
863\ldots 3086\ldots 632, & \quad 863\ldots 3266\ldots 632, \\
643\ldots 3266\ldots 654. & \quad 8k-times \quad 8k-times \quad 8k-times \quad 8k-times
\end{align*}
\]

Indicated number 4, at the end of the last number in this cycle, appears only for \(k \geq 1\).

For each \(k \leq 6\) this is the single 7-element minimal cycle of these Kaprekar’s transformations.

Fact 6. The following statements hold for every \(n \leq 20\).

If \(T_n\) possesses a cycle with the odd number of elements, then it possesses also a fixed point.

Moreover, we note that there exists \(n \leq 20\) such that the operator \(T_n\) possesses only the nontrivial orbits with the even numbers of elements, for example we may consider \(T_0, T_2, T_5\).

Fact 7. If \(a\) is an element belonging to the orbit of operator \(T_n\) composed of at least three numbers and \(a = \alpha_1\alpha_2\ldots\alpha_n\) and \(T_n(a) = \beta_1\beta_2\ldots\beta_n\) are the decimal representations of numbers \(a\) and \(T_n(a)\), respectively, then \(\alpha_k - \beta_k = \beta_{n-k+1} - \alpha_{n-k+1}\) for every \(k = 1, 2, \ldots, n\). For example, for the cycles of operator \(T_2\) (only two 4-element cycles are taken into account) we consider the following sequences of differences

\[
\beta_1 - \alpha_1, \quad \beta_2 - \alpha_2, \ldots, \quad \beta_5 - \alpha_5.
\]

Thus, for the cycle

\[
\begin{align*}
&62964 = a = T_5^1(a), \quad 71973 = T_5^2(a), \\
&83952 = T_5^3(a), \quad 74943 = T_5^4(a)
\end{align*}
\]

we have

\[
\begin{align*}
&-1, -2, 0, 2, 1: \quad T_5^2(a) - T_5^1(a) \\
&1, 0, -1, 0, -1: \quad T_5^3(a) - a
\end{align*}
\]

whereas for the cycle

\[
\begin{align*}
&61974, 82962, 75933, 63954
\end{align*}
\]

we have

\[
0, -2, 0, 2, 0; \quad 2, 1, 0, -1, -2; \quad -1, 3, 0, -3, 1; \quad -1, -2, 0, 2, 1.
\]

III. Conclusions

Although one can find quite a lot of references concerning the subject of the discussed here Kaprekar’s transformations (see the References in [1]), we have noticed yet several lacks in descriptions of the orbits of \(T_n\) transformations, even for \(n \leq 10\). Aim of our work was to complete these lacks, in which we succeeded, and we did even more. Our achievements have been indicated and included in Section II. One should emphasize especially the theorems concerning the possibility of “expanding” the fixed points and cycles of a given Kaprekar’s transformation \(T_n\), \(n \leq 18\), to the fixed points and cycles of infinitely many Kaprekar’s transformations (which, by the way, gives the answer to a question whether there exist infinitely many \(n \in \mathbb{N}\) such that \(T_n\) possesses a fixed point - similar fact concerns the possession of 3,5,7-element orbits). For our research we introduced several new concepts which, in the context of obtained numerical results, brought us to some theoretical results and conjectures. We derived some of our theorems and conjectures presented in Section II also for the generalizations of Kaprekar’s transformations (obeying the Q-Kaprekar’s transformation from [1] which will be the subject of the created now next paper. We intend also to use the experience, gained by applying the numerical results in theory, in didactic work by showing to the students the possibilities of seemingly simple calculations. We will also use in this field the experiences of other authors (see [2], [3]).

APPENDIX

Description of tables presenting the cycles of Kaprekar’s transformations \(T_n\)

The table is composed in the following way

– in the first row the value of index \(n\) of the Kaprekar’s transformation \(T_n\) is given,

– the second row presents the amount of minimal cycles of the given length of the given transformation \(T_n\) as well as the information whether the given transformation preserves the strong Sharkovsky’s order or the Sharkovsky’s order (see definitions 1 and 2 in [1]),

– the third row shows how many \(n\)-digit numbers is transformed by the given Kaprekar’s transformation \(T_n\) (after the finite number of steps) onto the respective minimal cycle of this transformation,

– in the successive rows the successive cycles from the third row (except the trivial one, that is the zero cycle) are associated with: the order types (it concerns only the cycles of length greater than 1, see the proper definition in [1]); the sum of digits of particular elements of the cycle, in case when these sums are identical, we include them only once; the digit types, and again, in case when they are identical, we include them only once; the longest increasing interval of the given cycle, the longest increasing subsequence of the given cycle, the longest decreasing interval of the given cycle and the longest decreasing subsequence of the given cycle.

REFERENCES

$n = 5$

1 fixed point, 1 cycle of length 2, 2 cycles of length 4; strong Sharkovsky’s order

<table>
<thead>
<tr>
<th>success</th>
<th>cycles</th>
<th>order type</th>
<th>sum of digits</th>
<th>digit type</th>
<th>longest incr. interval, subseq., longest decr. interval, subseq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>(1, 2)</td>
<td>27, 36</td>
<td>(10, 8, 9), (9, 18, 9)</td>
<td>2, 1, 2, 1</td>
<td></td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>(1, 4, 3, 2)</td>
<td>27</td>
<td>(10, 8, 9)</td>
<td>2, 2, 3, 3</td>
<td></td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>(1, 2, 4, 3)</td>
<td>27</td>
<td>(10, 8, 9)</td>
<td>3, 3, 2, 2</td>
<td></td>
</tr>
</tbody>
</table>

$n = 6$

3 fixed points, 1 cycle of length 7; Sharkovsky’s order

<table>
<thead>
<tr>
<th>success</th>
<th>cycles</th>
<th>order type</th>
<th>sum of digits</th>
<th>digit type</th>
<th>longest incr. interval, subseq., longest decr. interval, subseq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>(1, 5, 7, 6, 8, 4, 3, 2)</td>
<td>36</td>
<td>(10, 9, 8, 9)</td>
<td>2, 4, 4, 5</td>
<td></td>
</tr>
</tbody>
</table>

$n = 7$

1 fixed point, 1 cycle of length 8

<table>
<thead>
<tr>
<th>success</th>
<th>cycles</th>
<th>order type</th>
<th>sum of digits</th>
<th>digit type</th>
<th>longest incr. interval, subseq., longest decr. interval, subseq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>(1, 5, 7, 6, 8, 4, 3, 2)</td>
<td>36</td>
<td>(10, 9, 8, 9)</td>
<td>2, 4, 4, 5</td>
<td></td>
</tr>
</tbody>
</table>

$n = 8$

3 fixed points, 1 cycle of length 3, 1 cycle of length 7

<table>
<thead>
<tr>
<th>success</th>
<th>cycles</th>
<th>order type</th>
<th>sum of digits</th>
<th>digit type</th>
<th>longest incr. interval, subseq., longest decr. interval, subseq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1, \beta_2$</td>
<td>(1, 2, 3)</td>
<td>36</td>
<td>(10, 9, 9, 8)</td>
<td>3, 3, 1, 1</td>
<td></td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>(1, 5, 3, 4, 6, 7, 2)</td>
<td>36</td>
<td>(10, 9, 9, 8)</td>
<td>4, 5, 2, 3</td>
<td></td>
</tr>
</tbody>
</table>

$n = 9$

3 fixed points, 1 cycle of length 14; Sharkovsky’s order

<table>
<thead>
<tr>
<th>success</th>
<th>cycles</th>
<th>order type</th>
<th>sum of digits</th>
<th>digit type</th>
<th>longest incr. interval, subseq., longest decr. interval, subseq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>(1, 11, 10, 14, 9, 6, 3, 4, 2, 12, 5, 13, 8, 7)</td>
<td>45</td>
<td>(10, 9, 9, 8, 9)</td>
<td>2, 5, 4, 6</td>
<td></td>
</tr>
</tbody>
</table>
\begin{tabular}{|l|l|l|l|l|}
\hline
\textbf{successive cycles} & \textbf{order type} & \textbf{sum of digits} & \textbf{digit type} & \textbf{longest incr. interval, subseq., longest decr. interval, subseq.} \\
\hline
$\beta_1 - \beta_3$ & (1, 3, 2) & 45 & (10, 9, 9, 9, 8) & \\
$\beta_4$ & (1, 3, 2) & 45 & (10, 9, 9, 9, 8) & 2, 2, 2, 2 \\
$\beta_5, \beta_6$ & (1, 3, 2) & 45 & (10, 9, 9, 9, 8) & 2, 2, 2 \\
$\beta_7$ & (1, 3, 2) & 45 & (10, 9, 9, 9, 8) & 2, 2, 2 \\
$\beta_8$ & (1, 5, 3, 4, 6, 7, 2) & 45 & (10, 9, 9, 9, 8) & 4, 5, 2, 3 \\
\hline
\end{tabular}

\begin{tabular}{|l|l|l|l|l|}
\hline
\textbf{successive cycles} & \textbf{order type} & \textbf{sum of digits} & \textbf{digit type} & \textbf{longest incr. interval, subseq., longest decr. interval, subseq.} \\
\hline
$\beta_1$ & (1, 3, 2) & 54 & (10, 9, 9, 9, 8, 9) & \\
$\beta_2$ & (1, 5, 4, 3, 2) & 54 & (10, 9, 9, 9, 8, 9) & 2, 2, 4, 4 \\
$\beta_3$ & (1, 6, 4, 8, 2, 7, 5, 3) & 54 & (10, 9, 9, 9, 8, 9) & 2, 3, 3, 4 \\
\hline
\end{tabular}

\begin{tabular}{|l|l|l|l|l|}
\hline
\textbf{successive cycles} & \textbf{order type} & \textbf{sum of digits} & \textbf{digit type} & \textbf{longest incr. interval, subseq., longest decr. interval, subseq.} \\
\hline
$\beta_1$ & (1, 3, 2) & 54 & (10, 9, 9, 9, 8, 9) & \\
$\beta_2$ & (1, 5, 4, 3, 2) & 54 & (10, 9, 9, 9, 8, 9) & 2, 2, 4, 4 \\
$\beta_3$ & (1, 6, 4, 8, 2, 7, 5, 3) & 54 & (10, 9, 9, 9, 8, 9) & 2, 3, 3, 4 \\
\hline
\end{tabular}

\begin{tabular}{|l|l|l|l|l|}
\hline
\textbf{successive cycles} & \textbf{order type} & \textbf{sum of digits} & \textbf{digit type} & \textbf{longest incr. interval, subseq., longest decr. interval, subseq.} \\
\hline
$\beta_1$ & (1, 3, 2) & 72 & (10, 9, 9, 8, 18, 18) & \\
$\beta_{10}, \beta_7$ & (1, 3, 2) & 54 & (10, 9, 9, 9, 9, 8) & 2, 2, 2 \\
$\beta_8 - \beta_{11}$ & (1, 3, 2) & 54 & (10, 9, 9, 9, 9, 8) & 3, 3, 1, 1 \\
$\beta_{12}, \beta_{13}$ & (1, 3, 2) & 54 & (10, 9, 9, 9, 9, 8) & 2, 2, 2 \\
$\beta_{14}$ & (1, 3, 2) & 54 & (10, 9, 9, 9, 9, 8) & 3, 2, 1, 1 \\
$\beta_{15}$ & (1, 3, 2) & 54 & (10, 9, 9, 9, 9, 8) & 2, 2, 2 \\
$\beta_{16}$ & (1, 5, 3, 4, 6, 7, 2) & 54 & (10, 9, 9, 9, 9, 8) & 4, 5, 2, 3 \\
\hline
\end{tabular}
<table>
<thead>
<tr>
<th>successive cycles</th>
<th>order type</th>
<th>sum of digits</th>
<th>digit type</th>
<th>longest incr. interval, subseq., longest decr. interval, subseq.</th>
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<tbody>
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<td>$\beta_1$</td>
<td>(1, 2)</td>
<td>63</td>
<td>(10, 9, 9, 9, 9, 8, 9)</td>
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$n = 14$

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\[ n = 15 \]

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<tbody>
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<tr>
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